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COMMENT

Non-relativistic supersymmetry and gauge invariance

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Abstract. The gauge-invariance problem for a supersymmetric Schrödinger equation invariant under non-relativistic Euclidean supersymmetry is discussed. A gauge-invariant supermultiplet is considered which satisfies this equation. The equations of motion for its physical components are found to be relativistically invariant, which allows us to conclude that Lorentz invariance is a dynamical symmetry of the non-relativistic supersymmetric gauge problem.

Non-relativistic supersymmetry may be used as an alternative method for a unified description of relativistic boson and fermion fields. In a paper by Sokatchev and Stoyanov (1986) a supersymmetrisation of quantum mechanics models is proposed and studied, based on the supersymmetric extension of the three-dimensional Euclidean group rather than the Lorentz group (Gates *et al* 1983). A supersymmetric Schrödinger-like equation for a superfield wavefunction is defined there which possesses Euclidean invariance and is obviously non-relativistic: in it the time t is separated from the other coordinates of space like in ordinary quantum mechanics. The consequences of the non-relativistic supersymmetric equation for the physical components of the superfield wavefunction are that their equations of motion are the Lorentz-invariant Klein-Gordon and Dirac equations for the scalar and the spinor respectively. Thus Lorentz invariance appears as a dynamical symmetry of the considered non-relativistic supersymmetric quantum mechanics system. It has further been shown (Aneva *et al* 1987) that this important property of non-relativistic supersymmetry also survives in cases with an interaction. The original supersymmetric Schrödinger equation is redefined to contain different supersymmetrically invariant interaction terms, and it is found that it leads again to relativistic wave equations for particles interacting with external fields.

In the present comment we discuss the problem of gauge invariance of a quantum theory based on non-relativistic supersymmetry. We consider a superwavefunction that forms a representation of the non-relativistic superalgebra and that remains invariant under the action of an Abelian gauge transformation. Such a superfield contains within its components the electric and magnetic field strengths of electrodynamics. It satisfies the non-relativistic supersymmetric Schrödinger-like equation. We observe again that the equations of motion of the physical components of the superfield wavefunction are Lorentz invariant: they are the Weyl equation for the spinor field and the Maxwell equations for the electromagnetic field strengths. Therefore we conclude that Lorentz invariance appears as a dynamical symmetry of the non-relativistic gauge quantum system as well. In other words, we are able to obtain

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the 'on-shell' description of an O(3, 1) supersymmetric gauge multiplet starting with an O(3) supersymmetric theory.

For completeness, we record the supersymmetry algebra used in the following. Its even part consists of the three translation generators P_k and the three rotation generators I_k, $k=1,2,3$, satisfying the Lie algebra of the Euclidean group $T_3 \cdot O(3)$. The odd generators Q_α, $\alpha=1,2$, form an SU(2) complex spinor

$$[I_k, Q_\alpha] = -\tfrac{1}{2}(\sigma_k Q)_\alpha \tag{1}$$

and satisfy

$$\{Q_\alpha, Q_\beta\} = N(\sigma^k \varepsilon)_{\alpha\beta} P_k \tag{2}$$

where σ^k are the Pauli matrices and $\varepsilon = \mathrm{i}\sigma^2$ is the metric tensor in the spinor space $\varepsilon^2=-1$, $\varepsilon_{\alpha\beta} = -\varepsilon^{\alpha\beta} = -\varepsilon_{\beta\alpha}$. N is real and arbitrary.

A representation of the algebra is realised in the superspace (x_k, θ_α) where x_k are the coordinates of the three-dimensional Euclidean space and θ_α are two-component complex spinors, Grassmann variables:

$$\{\theta_\alpha, \theta_\beta\} = 0. \tag{3}$$

In the superspace (x_k, θ_α) the generators P_k and Q_α are represented by the differential operators

$$\begin{aligned} P_k &= -\mathrm{i}\frac{\partial}{\partial x^k} \\ Q_\alpha &= \mathrm{i}\frac{\partial}{\partial \theta^\alpha} + \mathrm{i}\frac{N}{2}(\sigma^k\theta)_\alpha P_k. \end{aligned} \tag{4}$$

In addition there exists a supercovariant differential operator

$$D_\alpha = \mathrm{i}\frac{\partial}{\partial \theta^\alpha} - \mathrm{i}\frac{N}{2}(\sigma^k\theta)_\alpha P_k \tag{5}$$

satisfying $\{D_\alpha, Q_\beta\}=0$ and

$$\{D_\alpha, D_\beta\} = -N(\sigma^k\varepsilon)_{\alpha\beta}P_k. \tag{6}$$

Hence

$$D_\alpha D^\beta D_\beta = N(\sigma^k D)_\alpha P_k. \tag{7}$$

The supersymmetric Schrödinger equation proposed by Sokatchev and Stoyanov (1986) for the wavefunction $\phi(t, x, \theta)$ (the time t is added merely as a parameter and does not transform under the superalgebra) has the form

$$K(x,\theta)\phi(t,x,\theta) = \mathrm{i}\frac{\partial}{\partial t}\phi(t,x,\theta). \tag{8}$$

The operator $K(x, \theta)$ constructed with the help of the supercovariant derivative

$$K(x,\theta) = \frac{4}{N^2}D^\alpha D_\alpha \tag{9}$$

plays the role of the supersymmetric kinetic energy operator.

After these preliminary comments we shall briefly sketch the gauge-invariance problem for the supersymmetric Schrödinger-like equation (8). Under an Abelian gauge transformation the superwavefunction $\phi(t, x, \theta)$ transforms as

$$\phi^{\mathrm{G}}(t,x,\theta) = \exp[-\mathrm{i}\Lambda(t,x,\theta)]\phi(t,x,\theta) \tag{10}$$

where the superfield $\Lambda(t, x, \theta)$ is the parameter of the U(1) gauge transformation. In the straightforward generalisation of this formula to a non-Abelian compact Lie group, $\Lambda(t, x, \theta)$ becomes a matrix superfield $\Lambda_{ij} = \Lambda_m J^m_{ij}$ where the matrices J^m are the Hermitian generators of the gauge group

$$[J^m, J^n] = \mathrm{i} t^{mnr} J^r$$

in the representation under which the scalar superfield ϕ_i transforms.

According to the well known scheme of a generalisation to superspace of the geometric approach to Yang-Mills theories (Gates *et al* 1980, Sohnius *et al* 1980, Wess and Bagger 1983) the three-dimensional Euclidean supersymmetric Yang-Mills theory is described by the gauge-potential superfields

$$X_A(t, x, \theta) = X^m_A J^m \qquad A = (\alpha, k) \tag{11}$$

and the corresponding Yang-Mills field strengths

$$F_{AB} = D_A X_B \mp D_B X_A + \mathrm{i}[X_A, X_B]_{\mp} - \mathrm{i} T^C_{AB} X_C \tag{12}$$

with $T^k_{\alpha\beta} = N(\sigma^k \varepsilon)_{\alpha\beta}$ and all other T terms zero. These field strengths are covariant under the gauge transformation

$$X_A \to \mathrm{e}^{-\mathrm{i}\Lambda}(X_A - \mathrm{i} D_A)\,\mathrm{e}^{\mathrm{i}\Lambda}. \tag{13}$$

To formulate a minimal manifestly supersymmetric gauge theory without superfluous fields one has to impose the gauge-covariant constraint

$$F_{\alpha\beta} = 0 \tag{14}$$

and solve the corresponding Bianchi identities

$$\sum_{\text{graded cyclic } A,B,C} (\nabla_A F_{BC} - \mathrm{i} T^D_{AB} F_{DC}) = 0 \tag{15}$$

subjected to this constraint. The symbol ∇_A denotes the gauge-covariant derivatives $\nabla_A \equiv D_A + \mathrm{i} X_A$. The result is a field strength spinor superfield $W_\alpha(t, x, \theta)$ obeying the gauge-covariant constraint equation

$$\nabla^\alpha W_\alpha = 0 \tag{16}$$

in terms of which the other field strengths are expressed:

$$F_{k\alpha} = (1/N)(\sigma^k W)_\alpha \qquad F_{kl} = (1/N^2)\varepsilon_{klm} \nabla^\alpha (\sigma^m W)_\alpha .$$

Since we are interested in the Abelian case we shall write the transformation (13) for the spinor gauge-connection superfield $X_\alpha(t, x, \theta)$ in a component form. Expanding the superfields $\Lambda(t, x, \theta)$ and $X_\alpha(t, x, \theta)$ in powers of θ_α:

$$\begin{aligned} &\Lambda(t, x, \theta) = g(t, x) + \theta^\alpha S_\alpha(t, x) + h(t, x)\theta^\beta \theta_\beta \\ &X_\alpha(t, x, \theta) = \psi_\alpha(t, x) + B(t, x)\theta_\alpha - (N/2)(\sigma^k \theta)_\alpha A_k(t, x) - (N/4)\chi_\alpha(t, x)\theta^\beta\theta_\beta \end{aligned} \tag{17}$$

we obtain

$$\begin{aligned} &\psi^{\mathrm{G}}_\alpha = \psi_\alpha + S_\alpha \qquad B^{\mathrm{G}} = B + 2h \\ &A^{\mathrm{G}}_k = A_k - \mathrm{i}\partial_k g \qquad \chi^{\mathrm{G}}_\alpha = \chi_\alpha - (\sigma^k \partial_k S)_\alpha . \end{aligned} \tag{18}$$

Among the components of $X_\alpha(t, x, \theta)$ there is a three-dimensional vector field $A_k(t, x)$ subjected to the usual gauge transformation—the addition of a gradient term. We therefore identify this field with the electromagnetic field.

From the form of the transformation (18) it is evident that we can choose a special gauge (analogous to that known in the literature as the Wess-Zumino gauge) in which the gauge-potential superfield X_α contains no other components but the vector A_k and the gauge-invariant spinor $U_\alpha = \chi_\alpha + \sigma^k \partial_k \psi_\alpha$, $\delta^G U_\alpha = 0$.

With the help of the gauge-potential superfield $X_\alpha(t, x, \theta)$ we can construct the corresponding supersymmetric field strengths, i.e. the spinor superfield W_α

$$W_\alpha(t, x, \theta) = D^\beta D_\alpha X_\beta(t, x, \theta) \tag{19}$$

which may also be written in the form

$$W_\alpha(t, x, \theta) = N(\sigma^k P_k X)_\alpha - D_\alpha D^\beta X_\beta.$$

It is readily seen that W_α remains invariant under the gauge transformation (13). We can very easily compute its components in the special gauge $X_\alpha = -(N/2)(\sigma^k\theta)_\alpha A_k + U_\alpha \theta\theta$. They are

$$W_\alpha(t, x, \theta) = U_\alpha(t, x) + \tfrac{1}{2}N^2 \varepsilon_{kln}(\sigma_n\theta)_\alpha P_k A_l(t, x) - \tfrac{1}{4}N(\sigma^l P_l U)_\alpha \theta\theta \tag{20}$$

and we recognise, in the form of the supermultiplet W_α given by (20), the most general solution of the Abelian constraint equation (16) $D^\alpha W_\alpha = 0$.

We postulate now that the gauge-invariant superfield W_α satisfies the supersymmetric Schrödinger-like equation

$$K(x, \theta) W_\alpha(t, x, \theta) = \mathrm{i} \frac{\partial}{\partial t} W_\alpha(t, x, \theta). \tag{21}$$

In component form this equation is

$$\frac{4}{N}(\sigma^k P_k U)_\alpha = \mathrm{i}\frac{\partial}{\partial t} U_\alpha \tag{21a}$$

$$-\frac{4}{N}(\sigma_n\sigma_m)_{\alpha\beta} P_m \varepsilon_{kln} P_k A_l = \mathrm{i}\frac{\partial}{\partial t} \varepsilon_{kln} P_k A_l (\sigma_n)_{\alpha\beta} \tag{21b}$$

$$-P_k^2 U_\alpha = -\mathrm{i}\frac{\partial}{\partial t}\frac{N}{4}(\sigma^l P_l U)_\alpha. \tag{21c}$$

We observe that the first of these equations is again the Lorentz-invariant Weyl equation for the spinor U_α:

$$\mathrm{i}(\sigma^\mu \partial_\mu U)_\alpha = 0$$

with $\sigma^\mu = (\sigma^0, \sigma^k)$ and $x^0 = (4/N)t$, $\partial_0 = \partial/\partial x^0$.

Equation (21c) needs no comment since it is the first equation squared. We are going to discuss the second equation (21b). Due to its matrix structure it is in fact a pair of equations for the complex vector $G_n \equiv \mathrm{i}\varepsilon_{kln}\partial_k A_l$:

$$\operatorname{div} G = 0 \tag{22a}$$

$$\partial_0 G + \mathrm{i} \operatorname{curl} G = 0. \tag{22b}$$

Making use of the quaternion language we observe that (21b) can be rewritten in a quaternion form for the matrix $\mathbb{G} = G_k \sigma_k$:

$$\underset{\sim}{\partial}\mathbb{G} = 0 \tag{23}$$

where $\underset{\sim}{\partial} = \partial_0 + \partial_i \sigma_i$. With the help of the definition $G_n = E_n + \mathrm{i}H_n$ (E_n and H_n being the electric and magnetic field strengths) we recognise in (23) the quaternion form of the Maxwell equations. More precisely, (as has been proven by Weingarten (1973)) relation (23), together with the quaternion-conjugation condition $\tilde{\mathbb{G}} = -\mathbb{G}$ fulfilled by the electromagnetic field matrix $\mathbb{G}$, is equivalent to the pair of free Maxwell equations of electrodynamics:

$$\begin{aligned} \operatorname{div} E &= 0 \qquad & \operatorname{curl} E &= -\partial_0 H \\ \operatorname{div} H &= 0 \qquad & \operatorname{curl} H &= \partial_0 E. \end{aligned} \tag{24}$$

We obtain as a result that our original non-relativistic supersymmetric Schrödinger-like equation, satisfied by the gauge-invariant superfield wavefunction W_α, leads to relativistic wave equations for the component fields, namely the Maxwell equations for the boson components and the Weyl equation for the spinor components. We observe again the appearance of a dynamical Lorentz symmetry which means, in fact, that the non-relativistic field strength superfield W_α corresponds to an 'on shell' version of a O(3, 1) supersymmetric free gauge theory.

To summarise, we have considered, in the framework of non-relativistic three-dimensional Euclidean supersymmetry, a gauge-invariant supermultiplet which may be viewed as a supersymmetrisation of the electromagnetic field strengths. It describes (together with the corresponding gauge potential supermultiplet) an Abelian free gauge theory in a manifestly supersymmetric fashion and satisfies the previously proposed non-relativistic supersymmetric Schrödinger-like equation. The consequences of this supersymmetric equation for the physical components of the gauge-invariant supermultiplet are the relativistic Maxwell equations for the boson components and the Weyl equation for the fermion components. We therefore conclude that Lorentz invariance is a dynamical symmetry of the gauge sector of the non-relativistic quantum system. Thus non-relativistic O(3) supersymmetric theory provides the 'on-shell' description of an O(3, 1) supersymmetric gauge-invariant multiplet.

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References

Aneva B, Katzarov I and Stoyanov D 1987 *Bulg. J. Phys.* **4** 389
Gates S J, Grizaru M, Rocek M and Siegel W 1983 *Superspace* (New York: Benjamin)
Gates S J, Stelle K S and West P C 1980 *Nucl. Phys.* B **169** 347
Sohnius M F, Stelle K S and West P C 1980 *Nucl. Phys.* B **173** 127
Sokatchev E and Stoyanov D 1986 *Mod. Phys. Lett.* A **1** 577
Weingarten D 1973 *Ann. Phys., NY* **76** 510
Wess J and Bagger J 1983 *Supersymmetry and Supergravity* (Princeton, NJ: Princeton University Press)